

An EPQ Model with Deterioration, Partial Backlogging, and Carbon Emissions: A Comparison of Two Minimization Methods

Vania Ilena Purwanto¹, Dharma Lesmono^{1,*}, Jonathan Hoseana¹

¹ Center for Mathematics and Society, Faculty of Science, Parahyangan Catholic University, Bandung 40141, Indonesia

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ABSTRACT

In 2025, Wahyudi et al. constructed an EPQ-type inventory model involving non-defective and defective products, a demand rate that depends quadratically on time, a holding-cost rate that depends linearly on time, deterioration, and carbon emissions from production and storage activities. In that study, the total cost minimization was conducted using a method involving Maclaurin polynomial approximation without an investigation of the result's accuracy. In the present study, we modify the inventory model constructed by Wahyudi et al. by removing the defective-product assumption and incorporating a partial backlogging mechanism. We divide each inventory cycle into three periods: the production-consumption period, the post-production consumption period, and the shortage period. We conduct the total cost minimization using not only the method involving Maclaurin polynomial approximation but also the Lagrange multiplier method. Across five numerical parameter sets, the two methods yield closely comparable minimizers, with relative differences in the minimum total cost of at most 0.00635%. Furthermore, a sensitivity analysis shows that the length of the production-consumption period, the length of the period after production ceases, the no-shortage time proportion, the total carbon emissions, the regular production quantity, and the shortage quantity depend most sensitively on the lost-sales cost, whereas the minimum total cost depends most sensitively on the initial demand rate.

1. Introduction

Effective inventory management is essential for manufacturing companies in order to ensure the availability of products while preventing excessive inventory-related expenses. To support such management, numerous mathematical models have been developed in order to assist the determination of appropriate production strategies. One such model is the Economic Production Quantity (EPQ) model, the simplest form of which was introduced by Taft in 1918 [1]. Taft's EPQ-type inventory model assumes a constant demand rate, a constant holding-cost rate, no product deterioration, and no inventory shortages. While these assumptions simplify the model's mathematical analysis, they do not fully reflect the operating conditions often encountered in practice.

* Corresponding author.

E-mail address: jdharma@unpar.ac.id

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Subsequent studies have thus proposed various modifications of Taft's EPQ-type model by incorporating factors that better reflect real-world operating conditions. One factor frequently incorporated is product deterioration —the decline in the product's quality during storage— which is particularly relevant to products with limited shelf lives, such as food, beverages, and medicines. Additionally, the constant demand rate assumption is often replaced with an assumption that the demand rate depends on time through a certain functional form, which can be chosen to represent situations in which demand depends on seasonality, market trends, or changes in consumer preferences. A functional form commonly used for this purpose is the quadratic polynomial form [2]. Likewise, the constant holding-cost-rate assumption may be replaced with an assumption that the holding-cost rate varies over time —for example, linearly— owing to increases in warehouse rental costs, energy costs, or maintenance expenses [2, 3]. A recent review by Karimi [4] identifies variable demand and cost functions, allowable shortages with partial backlogging, environmental considerations, and production models among promising directions for studies on deteriorating-inventory systems. Related recent developments also include fractional inventory formulations and fuzzy optimization approaches [5, 6], as well as fuzzy and neutrosophic inventory models [7–9].

More recently, inventory models have also begun to incorporate environmental considerations, particularly the carbon emissions generated by production and storage activities [10–12]. In 2019, Daryanto and Wee [13] developed an EPQ-type inventory model for a deteriorating product that accounted not only for carbon emissions but also for defective items resulting from an imperfect production process. In their model, the demand rate was assumed to be constant, as were the holding-cost rates for both non-defective and defective products. In 2025, Wahyudi *et al.*, [14] modified the model of Daryanto and Wee [13] by assuming that the demand rate depends quadratically on time and that the per-unit per-time-unit holding-cost rate for non-defective products depends linearly on time. In both works [13, 14], the company's total cost was formulated as a function of two decision variables subject to a single constraint, and the minimization of this function was conducted by first rewriting the constraint as an expression for one of the decision variables in terms of the other, so that the total cost could be rewritten as a function of only one decision variable. This process involved replacing an exponential expression containing the substituted decision variable with its second-degree Maclaurin polynomial approximation, but neither work [13, 14] investigated the accuracy of the resulting minimum total cost.

In this paper, we further modify the EPQ-type model of Wahyudi *et al.*, [14] in the following three respects. First, while the model of Wahyudi *et al.*, [14] accounts for defective items generated during the production process, we omit this component to focus specifically on deterioration as the mechanism through which product quality declines during storage. Second, the model proposed in this paper allows shortages and incorporates a partial-backlogging mechanism, whereby a proportion of unmet demand is backordered and the remainder is treated as lost sales [15]. Third, we conduct the total-cost minimization using not only the method involving Maclaurin polynomial approximation but also the Lagrange multiplier method. The latter method uses the exact form of the minimization constraint, thereby enabling an assessment of the accuracy of the former method. At the state where the minimum total cost is achieved, we also calculate the corresponding carbon emissions and conduct a sensitivity analysis of various key quantities in order to identify the parameters upon which the quantities depend most sensitively. Thus, the principal contribution of the present study lies on the use of an exact constrained minimization method —the Lagrange multiplier method— to assess the accuracy of a non-exact minimization method involving Maclaurin polynomial approximation in a non-trivial EPQ-type inventory model. More precisely, Table 1 highlights our contribution with respect to several previous works, while Figure 1 visualizes the methodological structure of our study.

Table 1

Selected related EPQ and inventory models and the methodological gap addressed in the present study

Study	Demand	Holding cost	Shortages	Carbon	Approximation validated
Daryanto and Wee [13]	Constant	Constant	Not modeled	Yes	No
Castellano and Glock [16]	Stochastic	Not central	Partial backlogging	No	No
San-José <i>et al.</i> , [11]	Power	Not central	Full backlogging	Yes	No
Wahyudi <i>et al.</i> , [14]	Quadratic	Linear	Not modeled	Yes	No
Present study	Quadratic	Linear	Partial backlogging	Yes	Yes

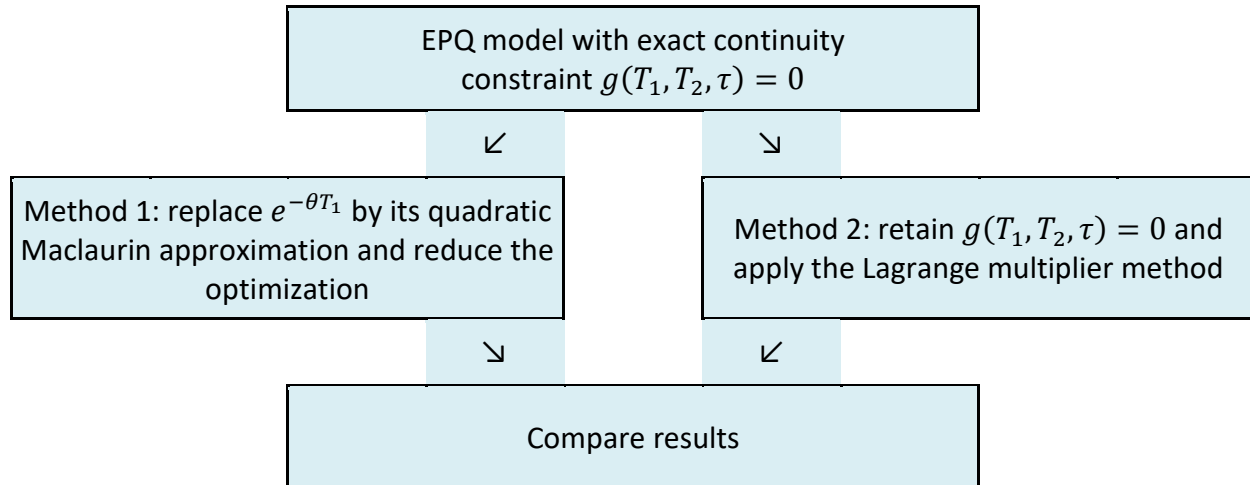


Fig. 1. Methodological structure of the study: the same EPQ model is solved using two different methods, after which the results are compared

The rest of the paper is organized as follows. In Section 2, we explain the construction of our EPQ-type inventory model, the formulation of the total inventory cost and carbon emissions, and the two total-cost minimization methods. In constructing the model, we first divide each inventory cycle into three periods: the production-consumption period, the post-production consumption period, and the shortage period. We derive the company's inventory level during these periods, adopting the standard convention that a negative inventory level represents backordered demand. These expressions yield a continuity constraint at the transition time $t = T_1$, at which the maximum inventory level is attained. Subsequently, we use the resulting quantities to formulate both the company's total inventory cost and total carbon emissions. In Section 3, we conduct a numerical simulation to illustrate the determination of the minimum total inventory cost using the two methods described in Section 2, as well as the associated total carbon emissions. We compare the results produced by the two methods in order to assess the accuracy of the method involving Maclaurin polynomial approximation for a baseline scenario and additional parameter sets. In addition, at the baseline minimum state obtained using the Lagrange multiplier method, we perform a sensitivity analysis of several key quantities to quantify their dependence on the model's parameters. In the final Section 4, we state our conclusions and describe avenues for further investigation.

2. Model construction

In this section, we construct a single-item EPQ-type inventory model for a deteriorating product by modifying the model proposed by Wahyudi *et al.*, [14]. Specifically, we incorporate product deterioration, time-dependent demand and holding-cost rates, carbon emissions from production and storage activities, and a partial-backlogging mechanism. We formulate the annual total cost incurred by a manufacturing company as a function of three decision variables: the length T_1 of the

production-consumption period, the length T_2 of the period after production ceases, and the proportion τ of that post-production period during which no shortage occurs. We then determine the values of these decision variables for which the total inventory cost achieves an interior local minimum. At the point (T_1, T_2, τ) where the local minimum is achieved, we calculate the total carbon emissions generated by the company's production and storage activities.

Our model is constructed under the following assumptions.

1. The inventory system is a single-product inventory system.
2. All parameters are deterministic and known with certainty.
3. The demand rate is a quadratic function of time and is non-negative throughout every feasible inventory cycle.
4. The holding-cost rate is a linear function of time.
5. Stored items deteriorate at a positive constant rate while the inventory level is positive, and deteriorated items are unrepairable.
6. During the production-consumption period, the production rate is sufficiently large that the inventory level increases until production ceases.
7. The demand rate is independent of inventory availability, meaning that demand continues even when a shortage occurs.
8. During a shortage, a constant proportion $\beta \in (0,1)$ of unmet demand is backordered and the remaining proportion $1 - \beta$ is treated as lost sales.
9. Backorders accumulated during a cycle are cleared at the cycle boundary through an external or emergency process before the next regular production-consumption period begins. The parameter c_b represents the aggregate per-unit backorder/clearance cost; any additional production, procurement, storage, and carbon emissions associated with this external clearance process lie outside the regular system boundary represented within the cycle.
10. No defective items are generated during the production process.
11. Carbon emissions are generated by production and storage activities.
12. The per-unit carbon-emission cost associated with production is the product of the average electrical energy used to produce one unit, the carbon-emission factor of electrical power generation, and the carbon tax rate.
13. The per-unit per-time-unit carbon-emission cost associated with storage is the product of the storage-space volume required by one unit, the average electrical energy used per unit volume of warehouse space per unit time, the carbon-emission factor of electrical power generation, and the carbon tax rate.

The notation used in our model is summarized in Table 2.

Table 2
Summary of the notation used in our EPQ-type inventory model

Symbol	Description	Unit
$D(t)$	demand rate that depends quadratically on time t	unit/year
P	production rate	unit/year
a	constant term in $D(t)$	unit/year
b	coefficient of t in $D(t)$	unit/year ²
c	coefficient of t^2 in $D(t)$	unit/year ³
θ	deterioration rate	year ⁻¹
$J(t)$	inventory level at time $t \in [0, T)$, with negative values representing backordered demand	unit
J_p	average on-hand inventory level over one cycle	unit
$J_{\max}$	maximum inventory level	unit
J_d	quantity of deteriorated inventory per cycle	unit
T	cycle length	year

Table 2
Continued

Symbol	Description	Unit
T_1	length of the production-consumption period	year
T_2	length of the period after production ceases	year
τ	proportion of the post-production period without a shortage	–
Q_p	quantity of items produced during one regular production run	unit
β	proportion of unmet demand that is backordered	–
Q_s	total unmet demand per cycle	unit
c_s	set-up cost per cycle	\$/cycle
c_p	production cost per unit	\$/unit
c_{pe}	carbon-emission cost generated by the production process per unit	\$/unit
$c_h(t)$	holding-cost rate that depends linearly on time t	\$/ (unit · year)
A	constant term in $c_h(t)$	\$/ (unit · year)
B	coefficient of t in $c_h(t)$	\$/ (unit · year ²)
c_{he}	carbon-emission cost generated by the storage process per unit per time unit	\$/ (unit · year)
c_d	deterioration cost per unit	\$/unit
c_b	backorder cost per unit	\$/unit
c_l	lost-sales cost per unit	\$/unit
e_p	average electrical energy used in the production process per unit	kWh/unit
e_w	average electrical energy used per unit volume of warehouse space per unit time	kWh/(m ³ · year)
v	volume of storage space occupied by one unit of the item	m ³ /unit
C_{TX}	carbon tax rate	\$/tonCO ₂
E_g	carbon-emission factor of electrical power generation	ton CO ₂ /kWh
SC	set-up cost per year	\$/year
PC	production-related cost per year	\$/year
HC	holding- and storage-emission-related cost per year	\$/year
DC	deterioration cost per year	\$/year
BC	backorder cost per year	\$/year
LSC	lost-sales cost per year	\$/year
TC	total cost incurred by the company	\$/year
TE	total carbon emissions generated by the company	ton CO ₂ /year

In this model, we divide each inventory cycle $[0, T)$ into three periods: the production-consumption period $[0, T_1)$, the post-production consumption period $[T_1, T_1 + \tau T_2)$, and the shortage period $[T_1 + \tau T_2, T_1 + T_2)$, where $T_1 > 0$, $T_2 > 0$, and $\tau \in [0, 1]$. The total cycle length is thus $T = T_1 + T_2$. During the production-consumption period, production and demand fulfilment occur simultaneously, causing the inventory level to increase until it reaches its maximum value $J_{\max}$ at $t = T_1$. Once production ceases, the post-production consumption period begins. During this period, no additional items are produced and the inventory level decreases owing to demand and deterioration until it reaches zero at $t = T_1 + \tau T_2$. The system then enters the shortage period. During this period, the inventory level is represented by a non-positive value whose absolute value equals the accumulated backordered demand. A proportion β of unmet demand is backordered, while the remaining proportion $1 - \beta$ is treated as lost sales. Figure 2 illustrates this convention.

Next, we define the quantity of items produced during one regular production run as

$$Q_p = PT_1, \quad (1)$$

and the demand rate as the following quadratic function of time $t \in [0, T)$:

$$D(t) = a + bt + ct^2. \quad (2)$$

Only decision-variable values satisfying

$$D(t) \geq 0 \quad (3)$$

for all $t \in [0, T)$ are regarded as feasible. We now derive the inventory level during each of the three periods.

1. In the production-consumption period $[0, T_1)$, the inventory level $\mathcal{I}_1(t)$ satisfies

$$\frac{d\mathcal{I}_1(t)}{dt} = P - D(t) - \theta\mathcal{I}_1(t), \quad \mathcal{I}_1(0) = 0.$$

Hence,

$$\mathcal{I}_1(t) = \frac{P-a}{\theta} + \frac{b(1-\theta t)}{\theta^2} - \frac{c(\theta^2 t^2 - 2\theta t + 2)}{\theta^3} + e^{-\theta t} \left(-\frac{P-a}{\theta} - \frac{b}{\theta^2} + \frac{2c}{\theta^3} \right). \quad (4)$$

To ensure that the inventory level increases throughout this period, feasible decision-variable values are further required to satisfy

$$P - D(t) - \theta\mathcal{I}_1(t) > 0 \quad (5)$$

for all $t \in [0, T_1)$.

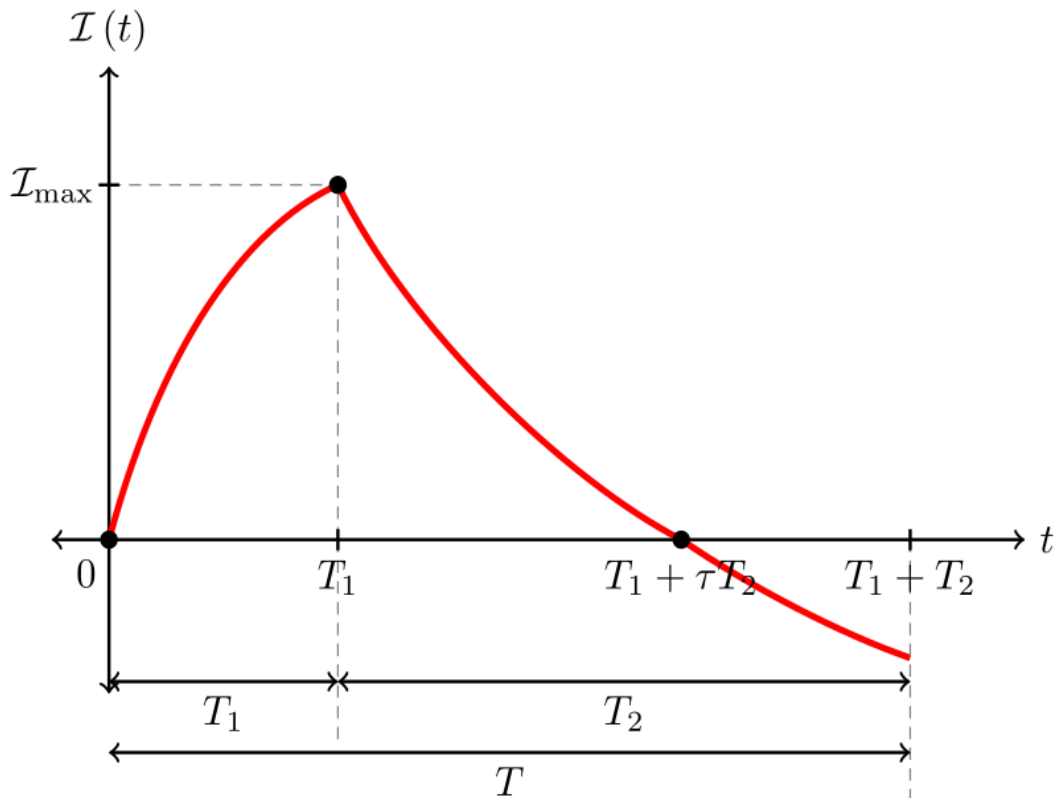


Fig. 2. The inventory level $\mathcal{I}(t)$ as a function of time $t \in [0, T)$, where $T = T_1 + T_2$. The right endpoint of the curve represents the inventory level immediately before backorder clearance

2. In the post-production consumption period $[T_1, T_1 + \tau T_2)$, the inventory level $\mathcal{I}_2(t)$ satisfies

$$\frac{d\mathcal{I}_2(t)}{dt} = -D(t) - \theta\mathcal{I}_2(t), \quad \lim_{t \rightarrow (T_1 + \tau T_2)^-} \mathcal{I}_2(t) = 0.$$

Its solution is

$$\mathcal{I}_2(t) = -\frac{a}{\theta} + \frac{b(1-\theta t)}{\theta^2} - \frac{c(\theta^2 t^2 - 2\theta t + 2)}{\theta^3} + e^{\theta(T_1 + \tau T_2 - t)} \left(\frac{a}{\theta} - \frac{b[1-\theta(T_1 + \tau T_2)]}{\theta^2} + \frac{c[\theta^2(T_1 + \tau T_2)^2 - 2\theta(T_1 + \tau T_2) + 2]}{\theta^3} \right). \quad (6)$$

3. In the shortage period $[T_1 + \tau T_2, T_1 + T_2)$, the inventory level becomes negative, with its absolute value representing accumulated backordered demand. Since the proportion β of unmet demand is backordered while the remaining proportion $1 - \beta$ is treated as lost sales, the inventory level $J_3(t)$ satisfies

$$\frac{dJ_3(t)}{dt} = -\beta D(t), \quad J_3(T_1 + \tau T_2) = 0.$$

Hence,

$$J_3(t) = -\beta \left[a(t - T_1 - \tau T_2) + \frac{b}{2}(t^2 - (T_1 + \tau T_2)^2) + \frac{c}{3}(t^3 - (T_1 + \tau T_2)^3) \right]. \quad (7)$$

The inventory level over one cycle $[0, T)$ can therefore be represented by the piecewise function

$$J(t) = \begin{cases} J_1(t), & \text{if } 0 \leq t < T_1; \\ J_2(t), & \text{if } T_1 \leq t < T_1 + \tau T_2; \\ J_3(t), & \text{if } T_1 + \tau T_2 \leq t < T_1 + T_2. \end{cases} \quad (8)$$

Continuity at $t = T_1$ requires

$$\lim_{t \rightarrow T_1^-} J_1(t) = J_2(T_1).$$

This condition can be written as $g(T_1, T_2, \tau) = 0$, where

$$g(T_1, T_2, \tau) = \frac{P}{\theta} + e^{-\theta T_1} \left(-\frac{P-a}{\theta} - \frac{b}{\theta^2} + \frac{2c}{\theta^3} \right) - e^{\theta \tau T_2} \left(\frac{a}{\theta} - \frac{b[1-\theta(T_1+\tau T_2)]}{\theta^2} + \frac{c[\theta^2(T_1+\tau T_2)^2 - 2\theta(T_1+\tau T_2) + 2]}{\theta^3} \right). \quad (9)$$

Under conditions (3) and (5), J_1 is increasing on $[0, T_1)$ and J_2 is decreasing on $[T_1, T_1 + \tau T_2)$. Hence, when (9) holds, the maximum inventory level is attained at $t = T_1$.

Next, the average on-hand inventory level over the cycle is

$$J_p = \frac{1}{T} \left[\int_0^{T_1} J_1(t) dt + \int_{T_1}^{T_1+\tau T_2} J_2(t) dt \right].$$

Using (4) and (6), we obtain

$$\begin{aligned} J_p = & \frac{1}{T} \left[\frac{(P-a)T_1}{\theta} + \frac{b}{\theta^2} \left(T_1 - \frac{\theta T_1^2}{2} \right) - \frac{c}{\theta^3} \left(\frac{\theta^2 T_1^3}{3} - \theta T_1^2 + 2T_1 \right) \right. \\ & + \frac{1-e^{-\theta T_1}}{\theta} \left(-\frac{P-a}{\theta} - \frac{b}{\theta^2} + \frac{2c}{\theta^3} \right) - \frac{a\tau T_2}{\theta} + \frac{b}{\theta^2} \left(\tau T_2 - \frac{\theta(2T_1\tau T_2 + \tau^2 T_2^2)}{2} \right) \\ & - \frac{c}{\theta^3} \left[\frac{\theta^2(3T_1^2\tau T_2 + 3T_1\tau^2 T_2^2 + \tau^3 T_2^3)}{3} - \theta(2T_1\tau T_2 + \tau^2 T_2^2) + 2\tau T_2 \right] \\ & \left. - \frac{1-e^{\theta \tau T_2}}{\theta} \left(\frac{a}{\theta} - \frac{b[1-\theta(T_1+\tau T_2)]}{\theta^2} + \frac{c[\theta^2(T_1+\tau T_2)^2 - 2\theta(T_1+\tau T_2) + 2]}{\theta^3} \right) \right]. \end{aligned} \quad (10)$$

The total unmet demand during the shortage period is

$$Q_s = \int_{T_1+\tau T_2}^{T_1+T_2} D(t) dt,$$

and therefore

$$Q_s = aT_2(1-\tau) + \frac{b}{2}[(T_1 + T_2)^2 - (T_1 + \tau T_2)^2] + \frac{c}{3}[(T_1 + T_2)^3 - (T_1 + \tau T_2)^3]. \quad (11)$$

The quantities of backordered and lost demand per cycle are therefore βQ_s and $(1 - \beta)Q_s$, respectively. In particular,

$$-\lim_{t \rightarrow (T_1+T_2)^-} J_3(t) = \beta Q_s. \quad (12)$$

Since deterioration affects the inventory only while the inventory level is positive, the quantity of items that deteriorate during one cycle follows from the material balance over the positive-inventory period:

$$J_d = Q_p - \int_0^{T_1 + \tau T_2} D(t) dt.$$

Thus,

$$J_d = PT_1 - \left[a(T_1 + \tau T_2) + \frac{b}{2}(T_1 + \tau T_2)^2 + \frac{c}{3}(T_1 + \tau T_2)^3 \right]. \quad (13)$$

This quantity equals the difference between the total regular production and the total demand fulfilled during the positive-inventory period.

The holding-cost rate is assumed to depend linearly on time according to

$$c_h(t) = A + Bt. \quad (14)$$

The last two carbon-emission assumptions lead to

$$c_{pe} = e_p E_g C_{TX} \quad \text{and} \quad c_{he} = v e_w E_g C_{TX}. \quad (15)$$

Because both $c_h(t)$ and the positive inventory level vary with time, the annual holding- and storage-emission-related cost is

$$HC(T_1, T_2, \tau) = \frac{1}{T} \left[\int_0^{T_1} (c_h(t) + c_{he}) J_1(t) dt + \int_{T_1}^{T_1 + \tau T_2} (c_h(t) + c_{he}) J_2(t) dt \right]. \quad (16)$$

Thus, the annual total inventory cost is the sum of the set-up cost $SC = c_s/T$, the production-related cost $PC = (c_p + c_{pe})Q_p/T$, the holding- and storage-emission-related cost (16), the deterioration cost $DC = c_d J_d/T$, the backorder cost $BC = c_b \beta Q_s/T$, and the lost-sales cost $LSC = c_l(1 - \beta)Q_s/T$. Therefore,

$$TC(T_1, T_2, \tau) = \frac{c_s}{T} + \frac{(c_p + c_{pe})Q_p}{T} + HC(T_1, T_2, \tau) + \frac{c_d J_d}{T} + \frac{c_b \beta Q_s}{T} + \frac{c_l(1 - \beta)Q_s}{T}. \quad (17)$$

Here, Q_p , J_p , Q_s , and J_d are given by equations (1), (10), (11), and (13), respectively.

The annual total carbon emissions generated by the company's regular production and storage activities are

$$TE(T_1, T_2, \tau) = \frac{e_p E_g}{T} Q_p + v e_w E_g J_p. \quad (18)$$

The quantities c_{pe} and c_{he} are monetary carbon-emission costs obtained by applying the carbon tax to physical emissions and are included in TC. By contrast, TE measures physical emissions in tons of CO₂ per year and is reported separately, as opposed to added again to the total cost. The clearance of backorders at the cycle boundary is outside the production and storage processes included in (18).

We conclude this section with a discussion of the two methods used in Section 3 to determine the triple (T_1, T_2, τ) that minimizes the company's total inventory cost. In both methods, only solutions satisfying $T_1 > 0$, $T_2 > 0$, $\tau \in [0, 1]$, and the feasibility conditions (3) and (5) are retained. In the first method, the exact continuity constraint $g(T_1, T_2, \tau) = 0$ is replaced by the approximate constraint (20), whereas in the second method the constraint $g(T_1, T_2, \tau) = 0$ is retained in its exact form.

1. In the first method, we apply the quadratic Maclaurin polynomial approximation

$$e^{-\theta T_1} \approx 1 - \theta T_1 + \frac{\theta^2 T_1^2}{2}.$$

The approximation error admits an immediate theoretical bound. Letting $x = \theta T_1 \geq 0$, Taylor's theorem gives

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{e^{-\xi} x^3}{6},$$

for some $\xi \in (0, x)$, and hence

$$\left| e^{-x} - \left(1 - x + \frac{x^2}{2} \right) \right| \leq \frac{x^3}{6}. \quad (19)$$

We thus rewrite the constraint $g(T_1, T_2, \tau) = 0$ as

$$0 = \frac{P}{\theta} + \left(1 - \theta T_1 + \frac{\theta^2 T_1^2}{2} \right) \left(-\frac{P-a}{\theta} - \frac{b}{\theta^2} + \frac{2c}{\theta^3} \right) - e^{\theta \tau T_2} \left(\frac{a}{\theta} - \frac{b[1-\theta(T_1+\tau T_2)]}{\theta^2} + \frac{c[\theta^2(T_1+\tau T_2)^2 - 2\theta(T_1+\tau T_2) + 2]}{\theta^3} \right).$$

This equation is equivalent to

$$\mathcal{X}(T_2, \tau) T_1^2 + \mathcal{Y}(T_2, \tau) T_1 + \mathcal{Z}(T_2, \tau) = 0, \quad (20)$$

where

$$\mathcal{X}(T_2, \tau) = \frac{\theta(a-P)-b}{2} + \frac{c(1-e^{\theta \tau T_2})}{\theta}, \quad (21)$$

$$\mathcal{Y}(T_2, \tau) = P - a + \frac{b(1-e^{\theta \tau T_2}) - 2c\tau T_2 e^{\theta \tau T_2}}{\theta} + \frac{2c(e^{\theta \tau T_2} - 1)}{\theta^2}, \quad (22)$$

$$\mathcal{Z}(T_2, \tau) = \frac{a(1-e^{\theta \tau T_2}) - b\tau T_2 e^{\theta \tau T_2} - c\tau^2 T_2^2 e^{\theta \tau T_2}}{\theta} + \frac{b(e^{\theta \tau T_2} - 1) + 2c\tau T_2 e^{\theta \tau T_2}}{\theta^2} + \frac{2c(1-e^{\theta \tau T_2})}{\theta^3}. \quad (23)$$

Assuming that $\mathcal{X}(T_2, \tau) \neq 0$ and $\mathcal{Y}(T_2, \tau)^2 - 4\mathcal{X}(T_2, \tau)\mathcal{Z}(T_2, \tau) > 0$, the quadratic formula gives the two differentiable branches

$$T_1^\pm = T_1^\pm(T_2, \tau) = \frac{-\mathcal{Y}(T_2, \tau) \pm \sqrt{\mathcal{Y}(T_2, \tau)^2 - 4\mathcal{X}(T_2, \tau)\mathcal{Z}(T_2, \tau)}}{2\mathcal{X}(T_2, \tau)}. \quad (24)$$

We retain only positive roots satisfying all feasibility conditions. For each of the two signs in (24), substituting the corresponding expression for T_1 into equation (17) yields a total-cost function of T_2 and τ , denoted by TC_+ or TC_- . For each $\pm \in \{+, -\}$, we solve

$$\frac{\partial \text{TC}_\pm}{\partial T_2} = 0, \quad \frac{\partial \text{TC}_\pm}{\partial \tau} = 0. \quad (25)$$

and retain only feasible solutions. We identify interior local minima using the second derivative test for functions of two variables [4, Thm. 13.8.6].

2. In the second method, we retain the equality constraint $g(T_1, T_2, \tau) = 0$ in its exact form. For interior feasible points, we apply the Lagrange multiplier method by defining

$$\mathcal{L}(T_1, T_2, \tau, \lambda) = \text{TC}(T_1, T_2, \tau) + \lambda g(T_1, T_2, \tau). \quad (26)$$

We then solve

$$\frac{\partial \mathcal{L}}{\partial T_1} = 0, \quad \frac{\partial \mathcal{L}}{\partial T_2} = 0, \quad \frac{\partial \mathcal{L}}{\partial \tau} = 0, \quad g(T_1, T_2, \tau) = 0. \quad (27)$$

for T_1, T_2, τ , and λ . We retain only feasible solutions and identify local minima using the second-order sufficient condition for equality-constrained optimization, namely positive definiteness of the Hessian of $\mathcal{L}$ with respect to (T_1, T_2, τ) on the tangent space of the constraint at the candidate point [28, Thm. 12.6].

The two methods differ only in the treatment of the continuity constraint. The first method converts that constraint into a quadratic relation for T_1 and then solves a two-variable stationary system, whereas the second method solves the four nonlinear equations in T_1 , T_2 , τ , and λ in (20). The former is therefore computationally lighter at the cost of a truncation error quantified by (13). In both methods, we check an appropriate second-order condition for local minimum after solving the first-order equations.

3. Numerical simulation and sensitivity analysis

In this section, we first compare the performance of the two methods described in the preceding section to determine the minimum total inventory cost in a baseline numerical scenario characterized by the parameter values presented in Table 3. Subsequently, we repeat the comparison using four additional parameter sets, thereby enabling an assessment across multiple scenarios. Additionally, at the state where the minimum total inventory cost is achieved, we conduct a sensitivity analysis of various key quantities in order to identify the parameters upon which these quantities depend most sensitively: the three decision variables, the minimum total inventory cost, the corresponding total carbon emissions, the regular production quantity, and the total unmet demand.

Table 3
Parameter values used in our numerical simulation

Parameter	Value	Unit
P	150,000	unit/year
a	20,000	unit/year
b	6,000	unit/year ²
c	−500	unit/year ³
θ	0.08	year ^{−1}
c_s	2,500	\$/cycle
c_p	0.3	\$/unit
A	0.1	\$/ (unit · year)
B	0.02	\$/ (unit · year ²)
c_d	0.25	\$/unit
c_b	0.3	\$/unit
β	0.4	—
c_l	0.55	\$/unit
v	0.001	m ³ /unit
C_{TX}	30	\$/tonCO ₂
e_p	0.8	kWh/unit
e_w	50	kWh/(m ³ · year)
E_g	0.85×10^{-3}	ton CO ₂ /kWh

3.1 Numerical simulation

Let us begin by applying our first method —the method involving Maclaurin polynomial approximation— to the numerical scenario represented by the parameter values in Table 3. For these parameter values, the demand function reads

$$D(t) = 20,000 + 6,000t - 500t^2$$

whose positive root is $6 + 2\sqrt{19} \approx 14.72$. Consequently, condition (2) requires $T_1 + T_2 \leq 6 + 2\sqrt{19} \approx 14.72$. Next, using formula (9), the carbon-emission cost generated by the production process per unit evaluates to

$$c_{pe} = e_p E_g C_{TX} = 0.8 \cdot 0.00085 \cdot 30 = 0.0204,$$

while the carbon-emission cost generated by the storage process per unit per time unit evaluates to

$$c_{he} = v e_w E_g C_{TX} = 0.001 \cdot 50 \cdot 0.00085 \cdot 30 = 0.001275.$$

Finally, the expressions $\mathcal{X}(T_2, \tau)$, $\mathcal{Y}(T_2, \tau)$, and $\mathcal{Z}(T_2, \tau)$ defined in equations (15), (16), and (17) read

$$\mathcal{X}(T_2, \tau) = 6,250 e^{0.08\tau T_2} - 14,450,$$

$$\mathcal{Y}(T_2, \tau) = 12,500\tau T_2 e^{0.08\tau T_2} - 231,250 e^{0.08\tau T_2} + 361,250,$$

$$\mathcal{Z}(T_2, \tau) = (6,250\tau^2 T_2^2 - 231,250\tau T_2 + 2,640,625)e^{0.08\tau T_2} - 2,640,625.$$

Substituting the expression (18) for $T_1^\pm = T_1^\pm(T_2, \tau)$ into the total inventory cost (11) gives the total cost expressions $TC_+ = TC_+(T_2, \tau)$ and $TC_- = TC_-(T_2, \tau)$. For the latter expression, which is visualized in Figure 3, no solution satisfying all feasibility conditions is obtained. Hence, only the former expression needs to be considered. Solving system (19) yields a point (T_2^*, τ^*) , where $T_2^* \approx 0.822675$ and $\tau^* \approx 0.966611$, at which $T_1^* \approx 0.147378$ and $T^* = T_1^* + T_2^* \approx 0.970053$. Since

$$\frac{\partial^2 TC_+}{\partial T_2^2} \cdot \frac{\partial^2 TC_+}{\partial \tau^2} - \left(\frac{\partial^2 TC_+}{\partial T_2 \partial \tau} \right)^2 \approx 9,000,807 > 0 \quad \text{and} \quad \frac{\partial^2 TC_+}{\partial T_2^2} \approx 8,747 > 0,$$

at this point the total inventory cost TC_+ achieves a local minimum. Furthermore, at this point, the total inventory cost (11), total carbon emissions (12), production quantity (1), and shortage quantity (7) evaluate to $TC^* \approx 11,407.4172$, $TE^* \approx 15.89521$, $Q_p^* \approx 22,106.63$, and $Q_s^* \approx 694.41$ respectively. A plot of the total inventory cost function near this minimum is presented in Figure 4. Figure 3 thus serves to document why the subsequent Maclaurin calculation proceeds with the plus branch only.

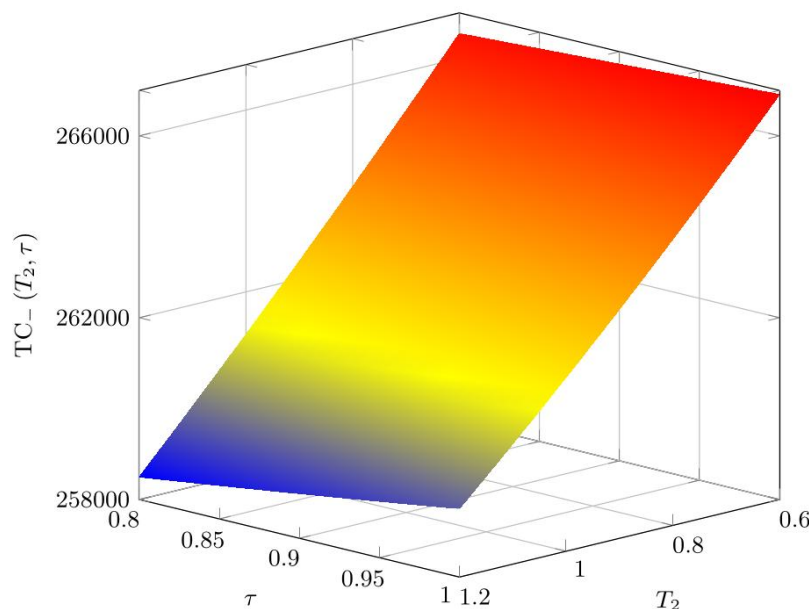


Fig. 3. The total inventory cost $TC_-(T_2, \tau)$ obtained from the minus branch T_1^- in (18), shown over a neighbourhood of the (T_2, τ) -values relevant to the baseline calculation. No feasible stationary solution from this branch is obtained

We next apply the second method —the Lagrange multiplier method— while retaining the exact constraint $g(T_1, T_2, \tau) = 0$. Solving system (20) yields

$$T_1^* \approx 0.147492, \quad T_2^* \approx 0.822241, \quad \tau^* \approx 0.967818, \quad \lambda^* \approx -0.588203,$$

and hence $T^* = T_1^* + T_2^* \approx 0.969733$. This solution satisfies all the feasibility conditions and lies in the interior of the feasible range of τ . Moreover, since $T^* \approx 0.969733 < 6 + 2\sqrt{19}$, condition (2) is satisfied strictly throughout the cycle, while condition (4) is also satisfied strictly on $[0, T_1^*)$. Hence, all inequality constraints are inactive at this point. To verify the second-order sufficient condition for equality-constrained optimization, the tangent space of the constraint at this point is

$$\{\mathbf{w} \in \mathbb{R}^3: \nabla g(T_1^*, T_2^*, \tau^*)^\top \mathbf{w} = 0\}.$$

Since

$$\nabla g(T_1^*, T_2^*, \tau^*) \approx \begin{pmatrix} 123,127.822 \\ -26,007.384 \\ -22,095.394 \end{pmatrix} \neq \mathbf{0},$$

a basis matrix for this tangent space is, for example,

$$\mathbf{Z} = \begin{pmatrix} 0.211223 & 0.179451 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

At the solution, the reduced Hessian of the Lagrangian is

$$\mathbf{Z}^\top \nabla_{(T_1, T_2, \tau)}^2 \mathcal{L}(T_1^*, T_2^*, \tau^*, \lambda^*) \mathbf{Z} \approx \begin{pmatrix} 8,757.108 & 5,037.836 \\ 5,037.836 & 3,924.174 \end{pmatrix}.$$

Its first leading principal minor is approximately $8,757.108 > 0$, while its determinant is approximately $8,984,622.697 > 0$. Hence, the reduced Hessian is positive definite. Therefore, by the second-order sufficient condition for equality-constrained optimization [18, Thm. 12.6], (T_1^*, T_2^*, τ^*) is a local minimum point. At this point,

$$TC^* \approx 11,406.6930, \quad TE^* \approx 15.91316, \quad Q_p^* \approx 22,123.86, \quad Q_s^* \approx 668.98,$$

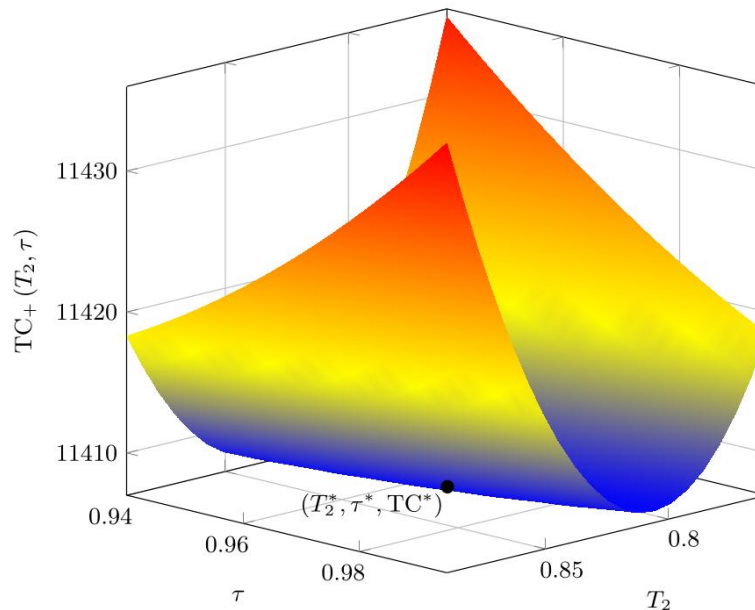


Fig. 4. The total inventory cost $TC_+(T_2, \tau)$ near the interior local minimum obtained using the method involving Maclaurin polynomial approximation. The marked point is $(T_2^*, \tau^*, TC^*) \approx (0.822675, 0.966611, 11,407.42)$

Table 4 compares these results with those obtained using the method involving Maclaurin polynomial approximation. The two methods produce very similar values of the decision variables and the resulting quantities. In particular, the minimum total costs obtained by the two methods differ by only approximately \$0.7242 per year, or 0.00635% of the value obtained using the Lagrange multiplier method. The largest relative difference among the quantities in Table 4 occurs for the shortage quantity, for which the difference is approximately 25.44 units, or 3.80%. For the former method, letting $x = \theta T_1^* \approx 0.0117902$, the inequality (13) gives an exponential-approximation error bound of $2.7316 \cdot 10^{-7}$. Evaluating the exact continuity constraint at that approximate solution gives $|g(T_1^*, T_2^*, \tau^*)| \approx 1.2298$, directly quantifying its constraint residual.

Table 4

Comparison of the results obtained using the method involving Maclaurin polynomial approximation and the Lagrange multiplier method

Quantity	Unit	Maclaurin	Lagrange	Absolute Difference
T_1^*	year	0.147378	0.147492	0.000115
T_2^*	year	0.822675	0.822241	0.000435
τ^*	–	0.966611	0.967818	0.001207
T^*	year	0.970053	0.969733	0.000320
TC^*	\$/year	11,407.4172	11,406.6930	0.7242
TE^*	ton CO ₂ /year	15.89521	15.91316	0.01795
Q_p^*	unit	22,106.63	22,123.86	17.23
Q_s^*	unit	694.41	668.98	25.44

To conduct the comparison beyond the baseline scenario, we repeated the application of both methods for four additional scenarios, each of which is characterized by the set of parameter values where only the parameter named in Table 5 was changed from Table 3, while all other parameters were held fixed. All five reported solutions satisfy both (2) and (4), lie in the interior of the range of τ , and satisfy the relevant second-order sufficient condition. Across these scenarios, the relative difference between the total costs at the two local minima is at most 0.00635%, while the largest relative difference among the three decision variables is below 0.19%. Moreover, $\theta T_{1,M}^* \leq 0.025931$, for which the right-hand side of (13) is at most $2.91 \cdot 10^{-6}$. Although these experiments do not establish universal accuracy, they show that the close agreement observed in the baseline scenario is also observed in the four additional scenarios.

Table 5

Comparison of the method involving Maclaurin approximation (results indexed by M) and the Lagrange multiplier method (results indexed by L) using five sets of parameter values. The relative total-cost difference is measured against the value obtained by the Lagrange multiplier method, and the rightmost column contains the largest relative difference among T_1^* , T_2^* , and τ^*

Case	$\theta T_{1,M}^*$	TC_M^*	TC_L^*	Relative TC difference (%)	$ g_M $	Maximum relative decision difference (%)
Baseline	0.011790	11,407.4172	11,406.6930	0.00635	1.2298	0.1247
$\theta = 0.20$	0.020092	11,831.0051	11,830.3924	0.00518	1.2441	0.1510
$\theta = 0.40$	0.025931	12,100.4282	12,099.9427	0.00401	1.0918	0.1868
$\beta = 0.60$	0.007044	10,971.0630	10,970.9561	0.00097	0.2626	0.0495
$c_l = 0.45$	0.006122	10,815.7836	10,815.7166	0.00062	0.1724	0.0413

3.2 Sensitivity analysis

Let us next analyze the sensitivity of the seven key quantities T_1^* , T_2^* , τ^* , TC^* , TE^* , Q_p^* , and Q_s^* evaluated at the local minimum state obtained using the Lagrange multiplier method, with respect

to the parameters involved in the model. We measure the sensitivity of each quantity Q to a parameter p using the sensitivity index

$$\gamma_p^Q = \frac{\partial Q}{\partial p} \cdot \frac{p}{Q},$$

which measures the relative change in Q resulting from a relative change in p [17, 18]. However, since explicit expressions for the quantities Q as functions of the parameters p are not available, we calculate the sensitivity indices γ_p^Q in an approximate manner [19].

Let $\Psi = \{-0.2\%, -0.1\%, 0.1\%, 0.2\%\}$. For each $\psi \in \Psi$, we denote by $\rho_{Q,p}(\psi)$ the percentage change in Q caused by a change of ψ in p . We define the approximate sensitivity index of Q with respect to p as

$$\bar{\gamma}_p^Q(\Psi) = \frac{1}{|\Psi|} \sum_{\psi \in \Psi} \frac{\rho_{Q,p}(\psi)}{\psi} = \frac{1}{4} \left[\frac{\rho_{Q,p}(-0.2\%)}{-0.2\%} + \frac{\rho_{Q,p}(-0.1\%)}{-0.1\%} + \frac{\rho_{Q,p}(0.1\%)}{0.1\%} + \frac{\rho_{Q,p}(0.2\%)}{0.2\%} \right].$$

A positive value of $\bar{\gamma}_p^Q(\Psi)$ indicates that the relative change in Q tends to have the same sign as the relative change applied to p , while a negative value indicates that the two relative changes tend to have opposite signs. Moreover, the magnitude $|\bar{\gamma}_p^Q(\Psi)|$ reflects the strength of the dependence of Q on p : a larger magnitude indicates greater sensitivity. The calculated values of $\bar{\gamma}_p^Q(\Psi)$ for each quantity $Q \in \{T_1^*, T_2^*, \tau^*, TC^*, TE^*, Q_p^*, Q_s^*\}$ and parameter p listed in Table 3 are presented in Table 6. In Table 6, the red cells indicate approximate sensitivity indices that are relatively large in absolute value, whereas the green cells indicate indices that are relatively small in absolute value.

Table 6

Approximate sensitivity indices of the quantities at the minimum state obtained using the Lagrange multiplier method with respect to the model parameters

p	Approximate sensitivity indices						
	$\bar{\gamma}_p^{T_1^*}$	$\bar{\gamma}_p^{T_2^*}$	$\bar{\gamma}_p^{\tau^*}$	$\bar{\gamma}_p^{TC^*}$	$\bar{\gamma}_p^{TE^*}$	$\bar{\gamma}_p^{Q_p^*}$	$\bar{\gamma}_p^{Q_s^*}$
P	-1.1869	0.3470	-0.3264	0.0219	-0.3005	-0.1869	10.1590
a	1.0451	-1.0039	0.9859	0.6558	1.7363	1.0451	-29.9138
b	0.1497	-0.8085	0.8059	0.1020	0.8135	0.1497	-24.8786
c	-0.0079	0.0803	-0.0802	-0.0057	-0.0749	-0.0079	2.4795
θ	-0.3024	0.2985	-0.5900	0.0399	-0.5172	-0.3024	18.0339
c_s	0.0000	1.3851	-1.3851	0.2260	-1.1745	0.0000	43.1539
c_p	-2.6323	4.1495	-6.3855	0.6000	-5.8083	-2.6323	196.2525
A	-0.6765	0.6262	-1.2007	0.0824	-1.1194	-0.6765	36.7221
B	-0.0732	0.0501	-0.1123	0.0060	-0.1062	-0.0732	3.4246
c_d	-0.1353	0.1252	-0.2401	0.0165	-0.2239	-0.1353	7.3444
c_b	0.9880	-1.7670	2.6061	0.0073	2.3576	0.9880	-80.1891
β	-0.8233	1.4725	-2.1717	-0.0060	-1.9647	-0.8233	66.8243
c_l	2.7169	-4.8594	7.1674	0.0199	6.4838	2.7169	-220.5216
v	-0.0086	0.0080	-0.0153	0.0011	0.0108	-0.0086	0.4682
C_{TX}	-0.1876	0.2901	-0.4495	0.0419	-0.4092	-0.1876	13.8133
e_p	-0.1790	0.2822	-0.4342	0.0408	0.5800	-0.1790	13.3451
e_w	-0.0086	0.0080	-0.0153	0.0011	0.0108	-0.0086	0.4682
E_g	-0.1876	0.2901	-0.4495	0.0419	0.5908	-0.1876	13.8133

The main observations are summarized in Table 7. One observes firstly that six of the seven reported quantities depend most sensitively on the lost-sales cost c_l . Moreover, increasing the lost-sales cost tends to lengthen the production-consumption period T_1^* , shorten the post-production period T_2^* , increase the no-shortage proportion τ^* , increase physical carbon emissions TE^* and the

regular production quantity Q_p^* , and reduce unmet demand Q_s^* . On the other hand, the total cost TC^* depends most sensitively on the initial demand rate a , an increase of which tends to increase the total cost. Furthermore, the very large relative sensitivity indices of Q_s^* require some care in interpretation. At the baseline exact solution, we compute that $T_2^*(1 - \tau^*) \approx 0.02646$ year and $Q_s^* \approx 668.98$ units, compared with $Q_p^* \approx 22,123.86$ units. Hence, the baseline shortage quantity is comparatively small and the solution lies close to the no-shortage regime $\tau = 1$, so that normalization by Q_s^* can amplify modest absolute changes into large relative indices. Re-solving every perturbed minimization problem at the stated tight tolerance reproduces these values, supporting their interpretation as genuine values of sensitivity index rather than indications of numerical instability.

Table 7

Parameters associated with the largest and smallest absolute approximate sensitivity indices of each quantity

Quantity	Most influential parameter (sign)	Least influential parameter(s)
T_1^*	$c_l(+)$	c_s
T_2^*	$c_l(-)$	v, e_w
τ^*	$c_l(+)$	v, e_w
TC^*	$a(+)$	v, e_w
TE^*	$c_l(+)$	v, e_w
Q_p^*	$c_l(+)$	c_s
Q_s^*	$c_l(-)$	v, e_w

4. Conclusions and future research

We have developed an EPQ model for a deteriorating product with a quadratic time-dependent demand rate, a linear time-dependent holding-cost rate, partial backlogging, and carbon emissions arising from production and storage activities. We divided each inventory cycle into a production-consumption period, a post-production consumption period, and a shortage period, in each of which we derived an expression for the time-dependent inventory level solving a first-order linear differential equation. We then used these expressions to formulate the total inventory cost, incorporating set-up, production, holding, deterioration, backorder, lost-sales, and carbon-emission costs, as well as the total carbon emissions generated by production and storage. We solved the resulting constrained minimization problem using two methods: a method involving quadratic Maclaurin polynomial approximation and the Lagrange multiplier method. The numerical results showed that the two methods produced closely comparable inventory policies not only in the selected baseline scenario but also in four additional test scenarios. In the baseline scenario, the former method yielded a minimum total cost of approximately \$11,407.42 per year and total carbon emissions of approximately 15.89521 ton CO₂ per year, while the latter method yielded a minimum total cost of approximately \$11,406.69 per year and total carbon emissions of approximately 15.91316 ton CO₂ per year. A sensitivity analysis in the baseline scenario further revealed that the lost-sales cost is the most influential parameter for the optimal production-consumption period, the post-production period, the no-shortage time proportion, the total carbon emissions, the regular production quantity, and the shortage quantity, whereas the minimum total inventory cost depends most sensitively on the initial demand rate.

Let us finish by describing avenues for further investigation. First, since all parameters in this study are assumed to be known with certainty, future studies could incorporate uncertainty in several parameters, such as those related to the demand and production rates, in order to better represent fluctuations often encountered in practice [20–22]. Likewise, the deterioration rate could be allowed to evolve over time so that changes in product quality or quantity during storage are represented

more realistically [23]. Moreover, the single-item setting could also be extended to multi-item or multi-product systems, possibly subject to storage-capacity, production-budget, or other resource constraints [24]. One could also incorporate lead time, potentially as a random variable, in order to model the delay between the time at which production is decided and that at which the inventory becomes available [25]. In addition, one could replace the constant partial-backlogging proportion used in the present model with a backlogging function that depends on waiting time [26]. Finally, the model considered in this paper could be extended by incorporating uncertainty in parameters such as demand, deterioration, backlogging, and carbon emissions using fuzzy, intuitionistic fuzzy, neutrosophic, hyperfuzzy, and plithogenic sets.

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Conflicts of Interest

The authors declare no conflicts of interest.

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